

An analytical solution versus half space BEM formulation for acoustic radiation and scattering from a rigid sphere

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Abstract. A half space problem in acoustics is described by introducing an infinite plane boundary that reflects the wave coming into the plane. A numerical solution using Boundary Element Method (BEM) has been known which is formulated using a modified Green's function in the Helmholtz Integral Formulation, which eliminates the discretization over the infinite plane. Hence, the discretization are confined to the body or obstacle in question only. This feature constitutes the main advantage of the BEM formulation for half space problems. However, no general analytical solution is available to verify the BEM results for half space problems. This paper is aimed to propose an analytical solution for the BEM to compare with, hence to verify the BEM calculation. This analytical approach is currently developed for a half space problem involving radiation and scattering of acoustic waves from a rigid sphere. The image of sphere as well as the image of the field point are defined with respect to the infinite plane. Then, an ad hoc solution is assumed involving a constant and the distance from the center of the sphere to the field point and the distance from the center of the image of the sphere to the field point. The constant is determined by imposing the boundary conditions. Test cases were run with several configuration involving the location of field points and the sphere. Comparison of the analytical solution with BEM calculations shows a good agreement between the two results..

1.Introduction

The Boundary Element Method (BEM) is one of the numerical methods that has been known to researchers in many area of research. The main advantage offers by the BEM is that it requires only the discretization of the boundary of the body in question. Therefore, in this scheme the dimension of the problem is reduced by one, e.g. a three dimensional problem is solved using a two dimensional calculation in the BEM. In acoustics, the application of the BEM includes many classes of problems involving radiation and scattering of acoustic waves from arbitrary shape bodies. Intensive developments in the BEM have been proposed by quite a number of researchers [1-9]. The acoustic medium may be confined in finite or infinite domain. Finite domain of acoustic media is found in so called interior problems in acoustics, while acoustic problems with infinite domain is known as exterior problems in acoustics. When the infinite domain is bounded by an infinite plane, the problem becomes a half space problem. The BEM formulation for half space problems uses a modified Green's function in the Helmholtz Integral Formulation and includes the image of the vibrating body in the radiation problem or the obstacle in the scattering problem. This formulation avoids the discretization of the infinite plane which simplifies the numerical calculation since it involves only the discretization of the radiating body in the radiation problem or the obstacle in the scattering problem. However, no



analytical solution available for half space problems to verify the BEM results. It is the aim of this paper to propose an analytical solution to verify the BEM calculation, specifically for radiation and scattering problems in a half space for a rigid sphere. Test cases were run for several configurations with respect to the field point locations and the distance between the center of the sphere and the infinite plane. Comparison of the results obtained from the analytical solutions with those from the BEM calculations show a very good agreement. Hence the BEM results are well verified.

2. The Half Space BEM Formulation

2.1. Radiation

The BEM formulation for acoustic radiation in a half space is given by the following equation [6,10] :

$$\int_{S_0} \left[\phi(Q) \frac{\partial \psi_H}{\partial \nu}(P, Q) - \psi_H(P, Q) \frac{\partial \phi}{\partial \nu}(Q) \right] dS(Q) = C(P) \phi(P), \quad (1)$$

where S_0 is the surface of the radiating object, ψ_H is the half space Green's function defined by

$$\psi_H(P, Q) = \exp[-ikR(P, Q)]/R(P, Q) + R_p \exp[-ikR'(P', Q)]/R'(P', Q), \quad (2)$$

where Q is any point on S_0 , P may on S_0 or outside of S_0 , k is the wave number ω/c where c is the speed of sound and ω is the frequency, and ν is the outward unit normal on S_0 , R_p is the reflection coefficient of the plane,

$$C(P) = 4\pi + \int_{S_0} \frac{\partial}{\partial \nu} \left[\frac{1}{R(P, Q)} \right] dS(Q), \quad (3)$$

2.2. Scattering

For scattering problem in a half space, the BEM formulation takes the form [6,10]:

$$\int_{S_0} \left[\phi_t(Q) \frac{\partial \psi_H}{\partial \nu}(P, Q) - \psi_H(P, Q) \frac{\partial \phi_t}{\partial \nu}(Q) \right] dS(Q) + 4\pi \phi'_i(P) = C(P) \phi_t(P), \quad (4)$$

where $\phi_t = \phi_s + \phi_i$ is the total velocity potential, ϕ_s is the scattered velocity potential, ϕ_i is the velocity potential of the incoming wave, $\phi'_i = \phi_i + \phi(P)$ is the sum of the velocity potential of the incoming wave and the velocity potential ϕ at P .

3. Analytical solution for radiation from a rigid sphere in a half space

Consider a uniformly pulsating sphere of radius a located with its center O at a distance B from an infinite rigid plane S_H as shown in Figure 1. The origin of the coordinate system is defined at the center of the sphere. Let O_1 be the center of the image of the sphere with respect to the plane. The velocity on the surface of the sphere is $U_a e^{i\omega t}$. Now assume an ad hoc solution:

$$\phi(r, t) = A \left[(1/r) e^{i(\omega t - kr)} + (1/r_1) e^{i(\omega t - kr_1)} \right] \quad (5)$$

where r is the distance from O to any field point P , r_1 is the distance from O_1 to P , and A is a constant.

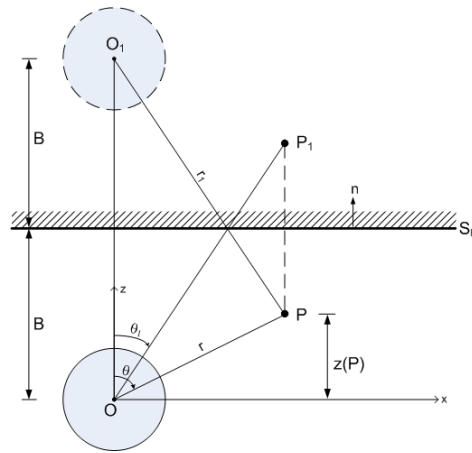


Figure 1 – Radiation from a pulsating sphere in a half space

On S_H :

$$\begin{aligned}\partial r / \partial n|_{S_H} &= \partial r / \partial z = \partial / \partial z (x^2 + y^2 + z^2)^{1/2} = z/r \\ \partial r_1 / \partial n|_{S_H} &= \partial r_1 / \partial z = \partial / \partial z [x^2 + y^2 + (z - 2B)^2]^{1/2} = (z - 2B)/r_1\end{aligned}$$

where n is the unit normal on S_H . Furthermore $r=r_1$ and $z=B$ on S_H . Therefore $\partial r / \partial n = B/r$ and $\partial r_1 / \partial n = -B/r$ so that

$$\partial r / \partial n|_{S_H} = -\partial r_1 / \partial n|_{S_H} \quad (6)$$

Then

$$\partial \phi / \partial n|_{S_H} = A \left[(-1/r^2 - ik/r) e^{i(\alpha t - kr)} \partial r / \partial n + (-1/r_1^2 - ik/r_1) e^{i(\alpha t - kr_1)} \partial r_1 / \partial n \right]_{S_H} = 0 \quad (7)$$

by virtue of Eq.(6). Thus the ad hoc solution (Eq.(5)) satisfies the boundary condition on S_H . The constant A can be determined from the boundary condition on the sphere, i.e.:

$$\begin{aligned}\nabla_r \phi|_{r=a} &= U_a \\ \text{or} \quad -\partial \phi / \partial r|_{r=a} &= U_a\end{aligned} \quad (8)$$

where the time dependent factor $e^{i\omega t}$ is suppressed for brevity. Express r_1 in terms of r and θ :

$$r_1^2 = r^2 + 4B^2 - 4Br \cos \theta \quad (9)$$

where θ is the angle between r and the z -axis. Then

$$\partial r_1 / \partial r = (r - 2B \cos \theta) / (r^2 + 4B^2 - 4Br \cos \theta)^{1/2} \quad (10)$$

Using Eq.(5) in Eq.(8) in light of Eq.(10) yields:

$$A = U_a a^2 / \left[(1 + ika) e^{-ika} + \left\{ (1 + ikaQ) / Q^2 \right\} \left\{ (1 - 2(B/a) \cos \theta) / Q \right\} e^{-ikaQ} \right] \quad (11)$$

where

$$Q = \left\{ 1 + 4(B/a)^2 - 4(B/a) \cos \theta \right\}^{1/2} \quad (12)$$

Incorporating Eq.(11) and Eq.(12) in Eq.(5) gives the final solution as follows:

$$\phi(r, t) = (a/r) \left[U_a a Q^3 \left\{ e^{-ikr} + \left(e^{-ikr\hat{Q}} / \hat{Q} \right) e^{i\alpha\hat{Q}} \right\} \right] / \left[(1 + ika) Q^3 e^{-ika} + (1 + ikaQ) \left\{ 1 - 2(B/a) \cos \theta \right\} e^{-ikaQ} \right] \quad (13)$$

where

$$\hat{Q} = \left\{ 1 + 4(B/r)^2 - 4(B/r) \cos \theta \right\}^{1/2} \quad (14)$$

Eq.(13) and Eq.(14) yield the analytical solution of the radiation of a pulsating rigid sphere in a half space with a rigid infinite plane. It may be noted that A is not a pure constant as assumed in Eq.(5), but $A = A(\cos \theta)$. However, taking into account that $\cos \theta \equiv \cos \theta_1$ in the vicinity of the plane, it can be shown that the boundary condition on the rigid plane S_H is satisfied.

4. Analytical solution for scattering from a rigid sphere in a half space

Suppose the sphere described in Fig.1 is not moving and is impinged by a plane wave $\phi_i = \phi_0 e^{i(kx + \omega t)}$ as shown in Fig.2. The incoming wave can be expanded into a series of spherical wave as [10]:

$$\phi_i = \phi_0 e^{i\omega t} \sum_{m=0}^{\infty} i^m (2m+1) P_m(\cos \delta) j_m(kr) \quad (15)$$

where $P_m(\cos \delta)$ are Legendre polynomials of order m , $j_m(kr)$ are spherical Bessel functions of order m and δ is the angle between r and the incoming wave. The image of the sphere has its center at O_1 , r_1 is the distance from O_1 to any field point P of interest.

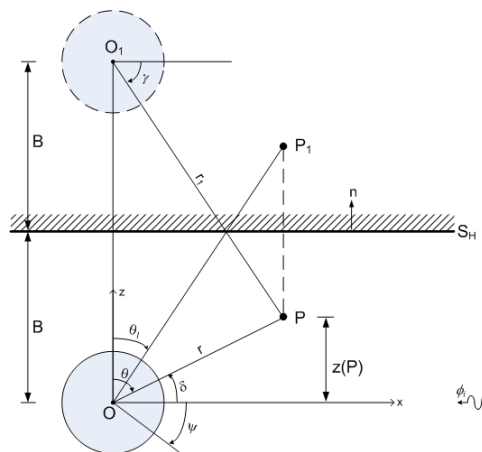


Figure 2 – Scattering from a rigid sphere in a half space

Assume the scattered potential to be given by

$$\phi_s = \sum_{m=0}^{\infty} [a_m P_m(\cos \delta) h_m(kr) + b_m P_m(\cos \gamma) h_m(kr_1)] e^{i\omega t} \quad (16)$$

where γ is the angle between r_1 and the direction of the incoming wave, $h_m(kr)$ are the Hankel functions of the second kind. Eq.(16) describes the scattered pressure as a series of spherical waves from the origin O and from O_1 . Each term in Eq.(16) is associated with the corresponding term in Eq.(15). The total potential ϕ_t is given by

$$\phi_t = \phi_i + \phi_s \quad (17)$$

Since the plane is rigid, the boundary condition on the plane is $-\nabla_z \phi_t|_{S_H} = 0$. Thus,

$$\partial/\partial z(\phi_i + \phi_s)|_{S_H} = 0$$

Drop the time dependent factor $e^{i\omega t}$ for brevity, $\partial\phi_i/\partial z|_{S_H} = \partial/\partial z(\phi_0 e^{ikx})|_{S_H} = 0$, so

$$\partial\phi_s/\partial z|_{S_H} = 0 \quad (18)$$

Using Eq.(16) in Eq.(18) and noting that for field points on the plane: $z=B$, $\delta=\gamma$ and $r=r_1$, it can be shown that $a_m = b_m$. Hence, Eq.(16) can be rewritten:

$$\phi_s = \sum_{m=0}^{\infty} a_m \{P_m(\cos \delta)h_m(kr) + P_m(\cos \gamma)h_m(kr_1)\}e^{i\omega t} \quad (19)$$

The boundary condition on the sphere is (the sphere is rigid) : $-\nabla_r \phi_t|_{r=a} = 0$ or

$$\partial\phi_i/\partial r|_{r=a} + \partial\phi_s/\partial r|_{r=a} = 0 \quad (20)$$

The m -th term of ϕ_i and ϕ_s must satisfy Eq.(20). Substituting Eq.(15) and Eq.(19) in Eq.(20) with little mathematical book keeping gives:

$$a_m = [\phi_0 i^m (2m+1)kP_m(\cos \delta)D_m(ka)\sin \delta_m(ka)]/[Term1 + Term2 + Term3] \quad (21)$$

where

$$Term1 = P_m(\cos \delta)kh'_m(ka) = -ikP_m(\cos \delta)D_m(ka)e^{-i\delta_m(ka)} \quad (22)$$

$$Term2 = P_m(\cos \gamma_0)(-ikD_m(kr_{10})e^{-i\delta_m(kr_{10})})\{1 - 2(B/a)\cos \theta\}/Q \quad (23)$$

$$Term3 = [m\{\cos \gamma_0 P_m(\cos \gamma_0) - P_{m-1}(\cos \gamma_0)\}/(\cos^2 \gamma_0 - 1)]\{\cos \delta(4(B/a)^2 - 2(B/a)\cos \theta)\}/(aQ^3)h_m(kr_{10}) \quad (24)$$

$\delta_m(ka)$ is the value of $\delta_m(kr)$ for $r = a$ in which

$\delta_m(kr)$ is defined such that $\frac{\partial h_m(kr)}{\partial(kr)} = D_m(kr)e^{-i\{\delta_m(kr)+\pi/2\}}$

and

$$D_m(kr) = \frac{1}{2m+1} [\{mj_{m-1}(kr) - (m+1)j_{m+1}(kr)\}^2 + \{mn_{m-1}(kr) - (m+1)n_{m+1}(kr)\}^2]^{1/2}$$

$j_m(kr)$ are the spherical Bessel functions with argument kr

$n_m(kr)$ are the spherical Neumann functions with argument kr

$$r_{10} = [4B^2 + a^2 - 4Bac\cos\theta]^{1/2}$$

Multiplying the numerator and the denominator by i , Eq.(21) becomes:

$$a_m = [k\phi_0 i^{m+1} (2m+1) P_m(\cos\delta) D_m(ka) \sin\delta_m(ka)] / [D_1 + D_2 + D_3] \quad (25)$$

where

$$D_1 = i\text{Term 1}, D_2 = i\text{Term 2}, D_3 = i\text{Term 3}$$

It may be noted that a_m does not look like a constant as assumed in Eq.(16), but $a_m = a_m(\cos\theta)$. However, it can be shown that $\partial(a_m)/\partial z|_{S_H} = 0$ which imply that $\partial\phi_s/\partial z|_{S_H} = 0$. Furthermore, it can also be shown that $\partial a_m/\partial z|_{S_H} = 0$. Thus a_m (Eq.(25)), when used in Eq.(19) will satisfy the boundary condition on S_H , i.e. $\partial\phi_t/\partial z|_{S_H} = 0$. Hence, the solution for the scattered potential can be expressed as:

$$\phi_s = k\phi_0 e^{i\alpha} \sum_{m=0}^{\infty} \{i^{m+1} (2m+1) P_m(\cos\delta) D_m(ka) \sin\delta_m(ka)\} \{P_m(\cos\delta) h_m(kr) + P_m(\cos\gamma) h_m(kr_1)\} / \{D_1 + D_2 + D_3\} \quad (26)$$

Equation (26) is the analytical solution of the scattering from a rigid sphere in a half space with rigid infinite plane.

5. Results and discussion

A number of test cases were run using the proposed analytical solution involving radiation and scattering of acoustic waves in a half space where comparison with the BEM calculations were examined. The infinite plane is assumed to be rigid. The object involved is a rigid sphere located at a variety of distance from the infinite plane. The first test case is a radiation from a pulsating rigid sphere. The center of the sphere is located at a distance $B=3a$ from the infinite plane. The sphere is vibrating with velocity on the surface equals to U_a . The boundary condition on the sphere is a uniform $U_a = \partial\phi/\partial\nu$ (pulsating sphere), and the normalized frequency of vibration of the sphere is $ka = 1$.

Figures 3 and 4 show the normalized acoustic pressure ($p=ikz_0\phi$) at distances of $3a$ and $2a$ from the center of the sphere, respectively as a function of polar angle plotted on a plane passing through the center of the sphere and perpendicular to the infinite plane. The plots of the two configurations show a good agreement between the analytical solutions and the BEM results.

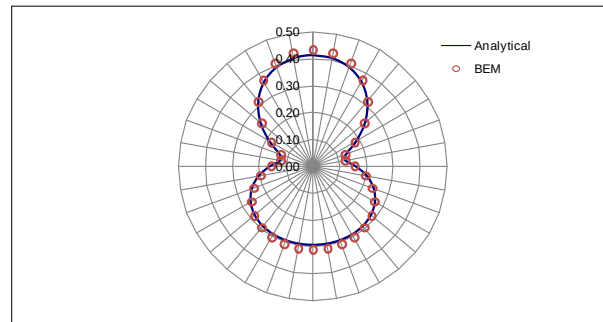


Figure 3 – Normalized radiated pressure at a distance $r = 3a$, when $ka = 1$, $B = 3a$

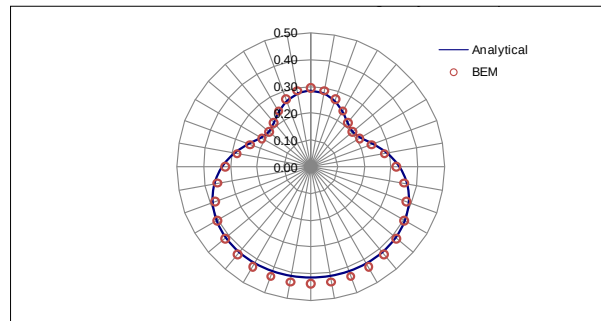


Figure 4 – Normalized radiated pressure at a distance $r = 2a$, when $ka = 1$, $B = 3a$

Next we locate the sphere closer to the plane such that $B=2a$ for the same frequency as the previous case, i.e. $ka=1$. Figure 5 shows the normalized radiated pressure plotted versus polar angle at distance $r=2a$. The BEM agrees with the analytical result to within 5% for this configuration.

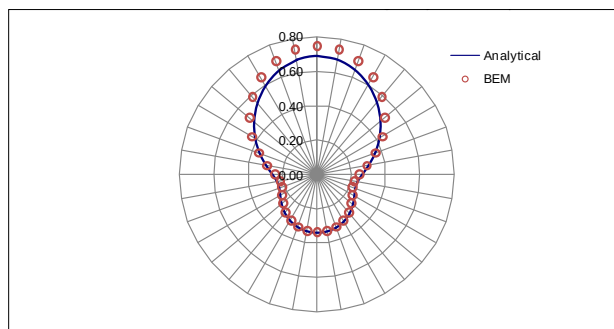


Figure 5 – Normalized radiated pressure at a distance $r = 2a$, when $ka = 1$, $B = 2a$

The next test case for scattering problem considered is the scattering from a rigid sphere of radius a located a distance $B=3a$ from the infinite rigid plane. The incident wave is a plane wave with a frequency of $ka=1$, propagating parallel to the rigid plane. Figures 6 and 7 are the polar plots of the scattered velocity potential, normalized by the velocity potential of the incident wave, plotted at a distance of $3a$ and $2a$ respectively from the center of the sphere. Both Figures 6 and 7 show good agreement between the BEM and the analytical results

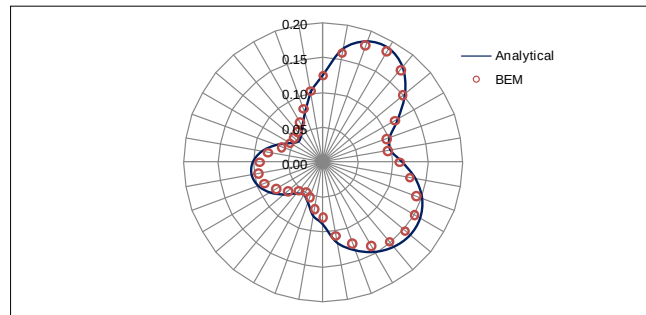


Figure 6 – Scattered velocity potential at a distance $r = 3a$, when $ka = 1$, $B = 3a$

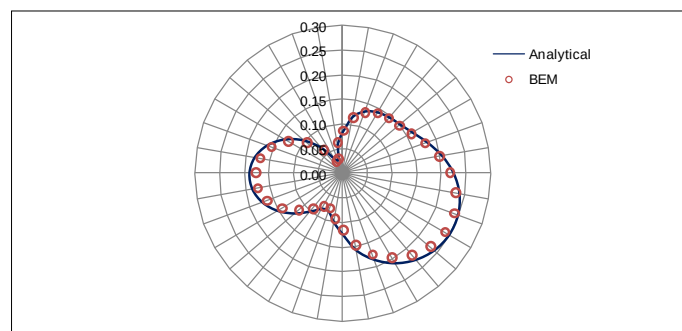


Figure 7 – Scattered velocity potential at a distance $r = 2a$, when $ka = 1$, $B = 3a$

The following test case demonstrates another configuration when the sphere is located closer to the rigid plane, i.e. $B=2a$. Figure 8 depicts the directivity pattern at a distance $r=2a$. It can be observed that the analytical solution and the BEM results agree very well.

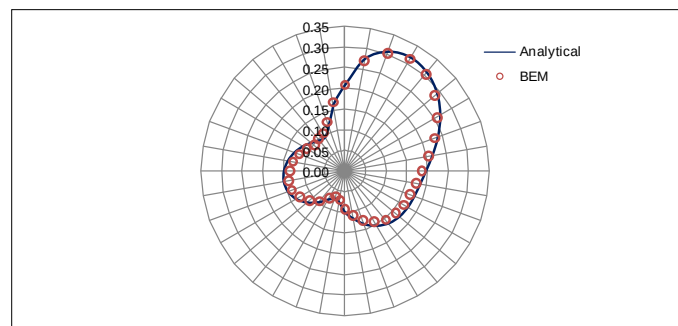


Figure 8 – Scattered velocity potential at a distance $r = 2a$, when $ka = 1$, $B = 2a$

In conclusion, an analytical solution has been derived for half space problems involving radiation and scattering of acoustic waves from a rigid sphere bounded by an infinite rigid plane. Verification has been demonstrated for various configuration taking varieties of the distance of the sphere from the infinite plane and the distance of the field point from the center of the sphere where the acoustic field is anticipated. Comparison with the BEM calculations shows a good agreement between the two results. At the same time the analytical solution may be used as a verification tool for numerical method calculations such as the BEM..

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